SCOPI Alumni

Trends and advances in optimization:

Industry applications with historical perspective

Tamás Terlaky

George N. and Soteria Kledaras '87 Endowed Chair Professor.

Chair, Department of Industrial and Systems Engineering

P.C. Rossin College of Engineering and Applied Science

Lehigh University, Bethlehem, PA

March 12, 2015, Veszprém

Trends and advances in optimization:

Theory ⟺ Computational methods ⟺ Computers

I) Solve larger problems faster -- New algorithm paradigms

II) New model classes

III) Software for Conic Linear Optimization

Industry applications

I) Optimization Ubiquitous

II) General and sector specific modeling/optimization tools

III) Not only industry, everywhere in life

Conclusions

SCOPI Alumni

I) Foundation: Linear Optimization (LO)

Duality and optimality are key tools in developing algorithms

Standard form for Linear Optimization (LO) Primal problem:

$$\begin{array}{rl} \min & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

where *A: m*x*n* has full row rank.

$$\begin{array}{rl} \max & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

$$A^T y + s = c, \quad s \geq 0$$

Weak duality: $c^T x = (A^T y + s)^T x = y^T b + s^T x \geq b^T y$

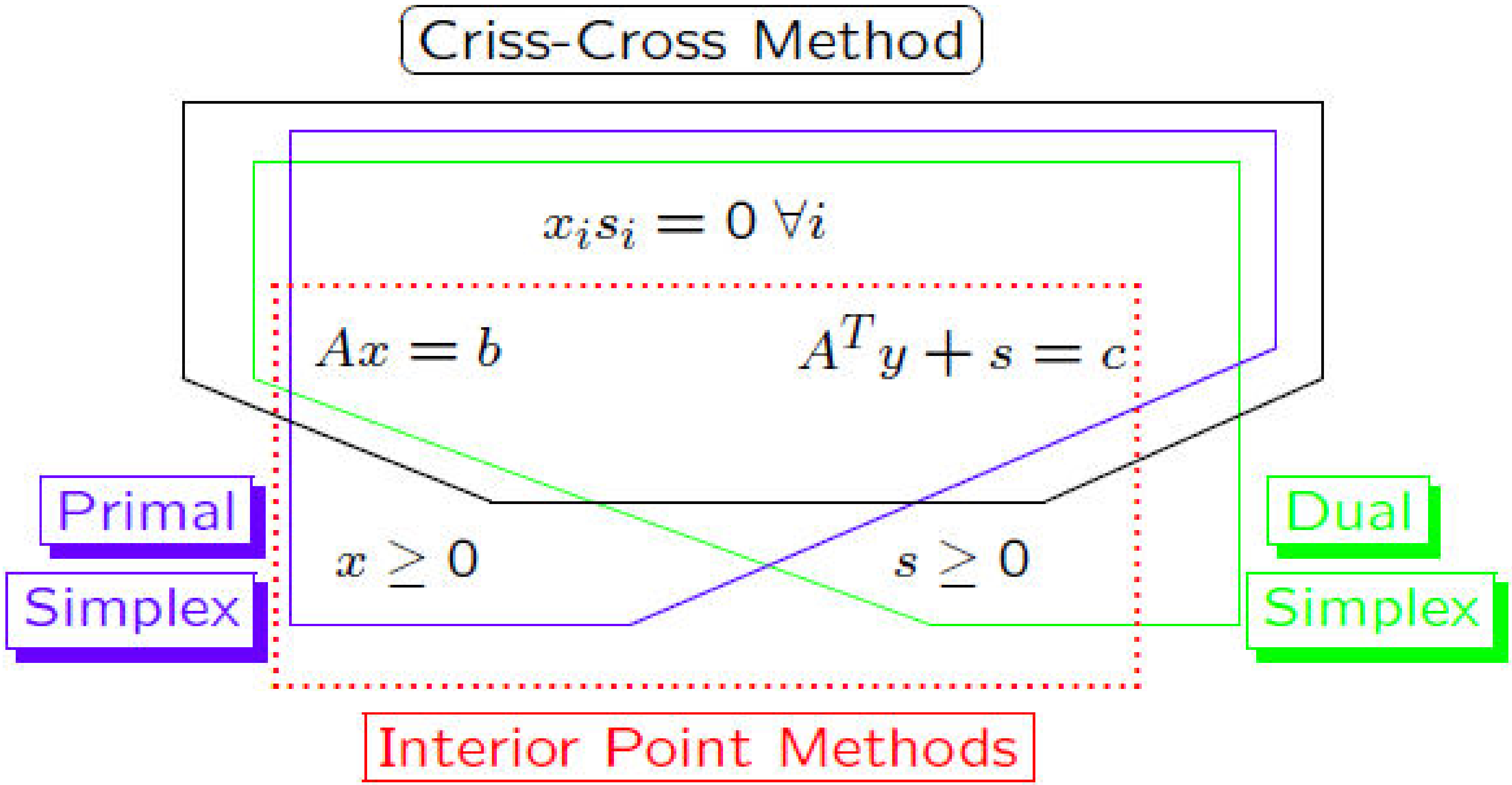
Optimality conditions:

$c^T x = b^T y$ or $x^T s = 0$ or $xs = 0 \Leftrightarrow x_i s_i = 0 \;\; \forall i$

SCOPI Alumni

Foundations of Algorithms for LO and QO

Primal feasibility, Dual feasibility, Complementarity

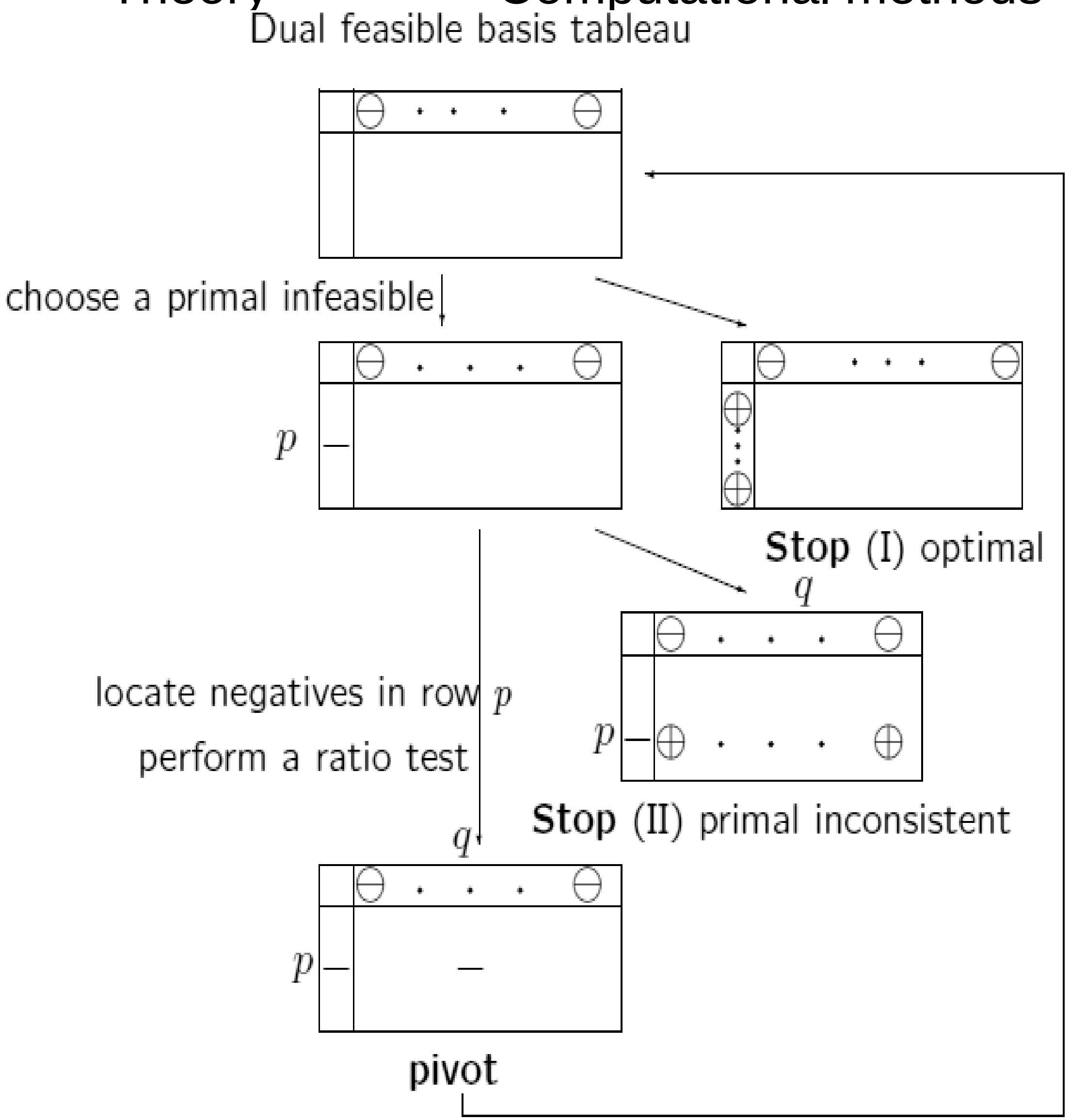


Algorithms keep a part of the optimality conditions while working towards satisfying the others

SCOPI Alumni

Simplex Algorithms – Dual Simplex

Theory | Computational methods | Computers



- Objective is monotone
- **Optimal Basis Solution**
- ***Issue: Degeneracy***
- **Finite variants**
- **Exponential** in the worst case – see Klee-Minty Cube
- **Efficient in practice**
- **“Average” and “expected” # of pivots is linear in *n***
- Activ research area

Interior Point Methods

Analytic center, central path and complexity

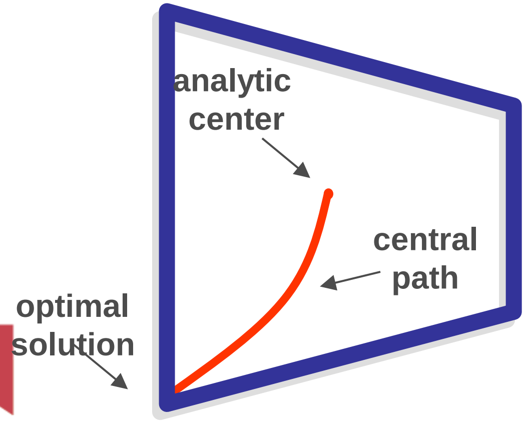
- **The central path start from the analytic center**
- **IPMs follow the central path**
- **converge to an optimal solution.**
- **IPMs are polynomial time algorithms for linear optimization**

$O(nL)$ **: number of iterations**

d **: number of inequalities**

L **: input-data bit-length**

μ **: central path parameter**



$$\max \quad b^{\mathrm{T}} y + \mu \sum_i \ln(c - A^{\mathrm{T}} y)_i$$

$$A^{\mathrm{T}} y \leq c$$

SCOPI Alumni

Interior Point Methods

- **Polynomial Complexity** depending on ***n*** and ***L***
- ***Iteration Complexity Bound Sharp***
- **Degeneracy is not an issue**
- **Redundancy (large *n*) may cause serious problems**

  Large *L* may cause extremely curly long path

  The central path is analytical – not geometrical
- **The central path converges to the analytical center of the optimal face**.
- IPMs produce **Exact Strictly complementary** solution

  Polynomial # of iterations followed by a

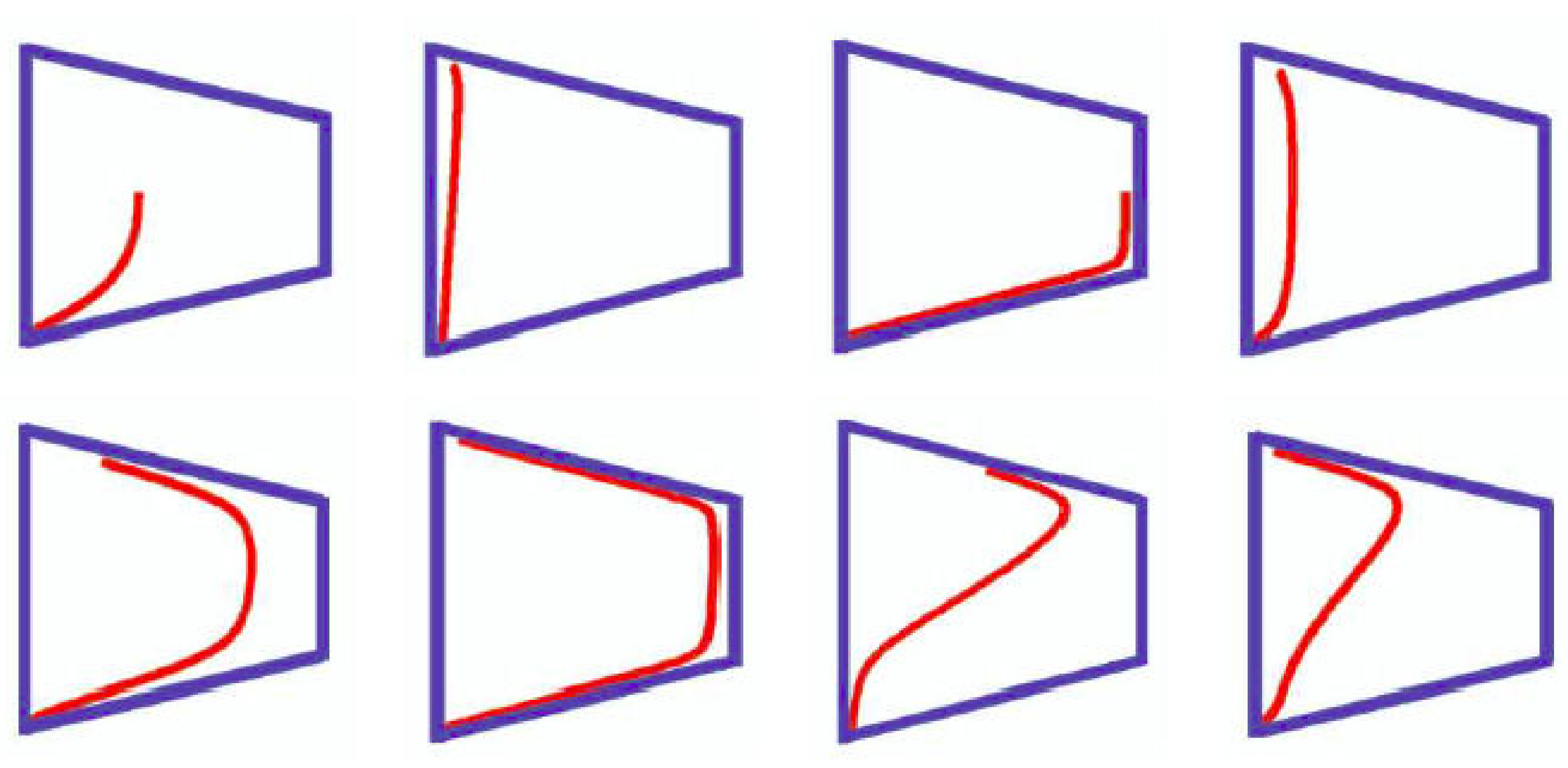
  Strongly polynomial rounding procedure
- From the exact strictly complementary solution pair an Optimal Basis can be obtained by a

  Strongly Polynomial Basis Identification Procedure

SCOPI Alumni

Central Path – with redundant representation

The central path is analytical, not geometrical!



Be Careful with modeling!

Ill formulated models are difficult to solve!

How curly the central path can be?

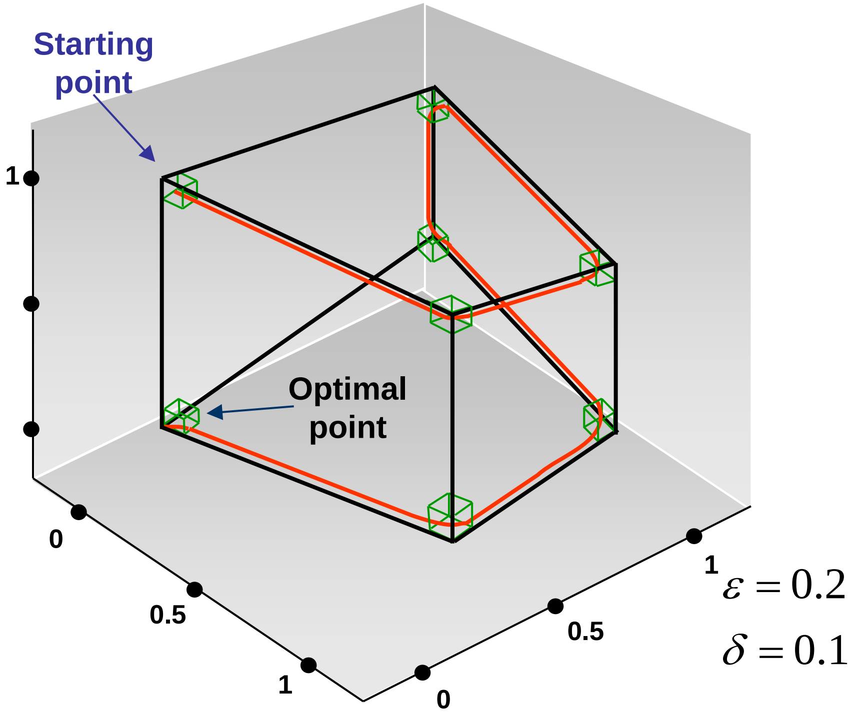
Note: **The central path depends on the representation of the feasible set;**
It is an analytic, not a geometric object.

Q: **Can the central path be bent along the edge-path followed by the simplex method on the Klee-Minty cube?**
(can the central path visit an arbitrary small neighborhood of all 2^n vertices?)

Yes! - if

we	**carefully**	**add**
an	**exponential**	**number**
of	**redundant**	**constrains**

IPMs iteration complexity bound is tight!



Solvers improve, enhanced by computer power

In a decade 1000 times better both computers and LO solvers

From: Bixby: Solving Real-World Linear Programs a decade and More of Progress

Instance	CPLEX 1.0	CPLEX 5.0 Dual	CPLEX 7.1 Primal	CPLEX 7.1 Dual
pds100	–	50413.1	2414.8	256.3
pds90	–	59981.0	2452.2	320.3
pds80	–	42055.4	2201.5	304.4
pds70	335292.1	21120.4	1504.1	197.8
pds60	205798.3	7442.6	852.4	160.5
pds50	122195.9	8509.9	493.2	114.6
pds40	58920.3	2816.8	188.3	79.3
pds30	15891.9	1154.9	74.8	39.1
pds20	5168.8	232.6	27.9	20.9
pds10	208.9	13.0	3.7	2.6
pds06	26.4	2.4	1.4	0.9
pds02	0.4	0.1	0.1	0.1

1979 DKV Százhalombatta

Size: 800x1100, IBM 360 with 128KM memory, Punch card MPS file

Solution time: about 3 hours by primal simplex

SCOPI Alumni

What is best? Simplex or Interior Point Methods

(Very) Large scale, degenerate: IPMs win, or the only option

Medium scale: depending on Structure

Re-optimization, warm start: Simplex wins

Table 11: **Solution times–Best of three**

Model	CPLEX 1.0	CPLEX 2.2	CPLEX 5.0	CPLEX 7.1	Algorithm
car	1555.0	203.0	117.1	67.3	barrier
continent	364.7	110.5	99.5	46.7	primal
energy1	1217.4	46.5	31.5	22.4	barrier
energy2	10130.1	171.4	71.7	32.4	barrier
energy3	21797.1	152.6	113.4	82.2	barrier
fuel	5619.5	999.1	340.5	124.7	barrier
initial	3832.2	102.2	51.3	15.5	dual
schedule	152404.0	252.3	132.0	47.9	barrier

SCOPI Alumni

II) New Model Classes

Conic, integer, black-box ….

Traditional model classes:

- LO, QO, MILO, Networks, …
- Convex, Nonlinear

Recent hot areas:

- Conic Linear Optimization
  - Second Order Cone Optimization (SOCO)
  - Semidefinite Optimization (SDO)
  - MISOCO and MISDO
- Mixed Integer Nonlinear Optimization
- Black-Box or Derivative Free Optimization (DFO)
- Simulation (based) optimization
- PDE based optimization

SCOPI Alumni

Conic Linear Optimization

Constraints are given as linear functions and convex sets

Primal-dual pair of CLO problems is given as

$$(P)\quad \min \quad c^T x \qquad\qquad (D)\quad \max \quad b^T y$$

$$\text{s.t.}\quad Ax - b \in \mathcal{C}_1 \qquad\qquad \text{s.t.}\quad c - A^T y \in \mathcal{C}_2^*$$

$$x \in \mathcal{C}_2 \qquad\qquad y \in \mathcal{C}_1^*,$$

where $b, y \in \mathbb{R}^m$, $c, x \in \mathbb{R}^n$, $A : m \times n$ matrix, $\mathcal{C}_1, \mathcal{C}_2$ are convex cones and $\mathcal{C}_i^* = \{s \in \mathbb{R}^n : x^T s \geq 0, \forall x \in \mathcal{C}_i\}$ are the dual cones for $i = 1, 2$.

These are solvable efficiently (in polynomial time) by using Interior Point Methods. LO is based on polyhedral cones.

Are all convex cones good???

SCOPI Alumni

Second Order Cone Optimization (SOCO)

Ice cream / Lorenz / second Order Cone

The second order cone in $\mathbb{R}^n$ is defined as

$$\mathcal{S}_2^n := \left\{ x \in \mathbb{R}^n : \sqrt{\sum_{i=1}^{n-1} x_i^2} \leq x_n \right\}.$$

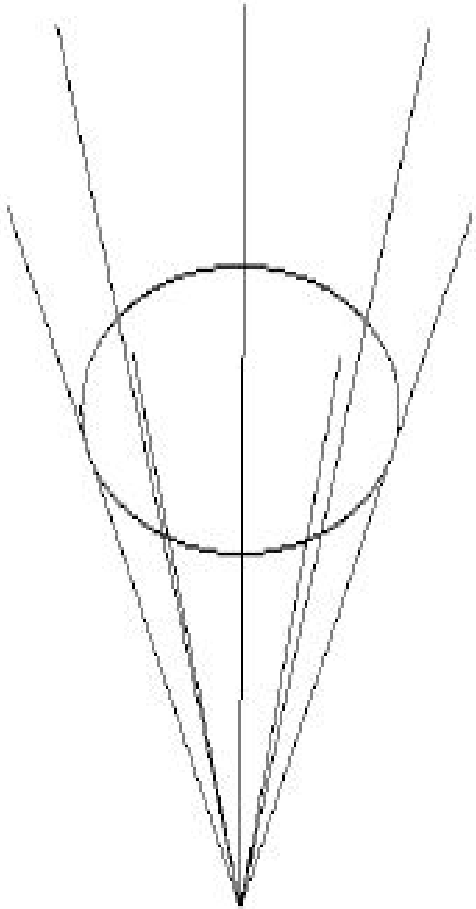
The name "ice cream cone" is coming from the 3-dimensional shape of the cone.

The second order cone is self-dual: $(\mathcal{S}_2^n)^* = \mathcal{S}_2^n$.

Optimization problems where the cones $\mathcal{C}_1$ and $\mathcal{C}_2$ are polyhedral and second order cones are *second order cone optimization(SOCO)* problems.

Significance

Norm minimization, robust optimization.



The ice-cream cone L^3

Alumni

Semidefinite Optimization

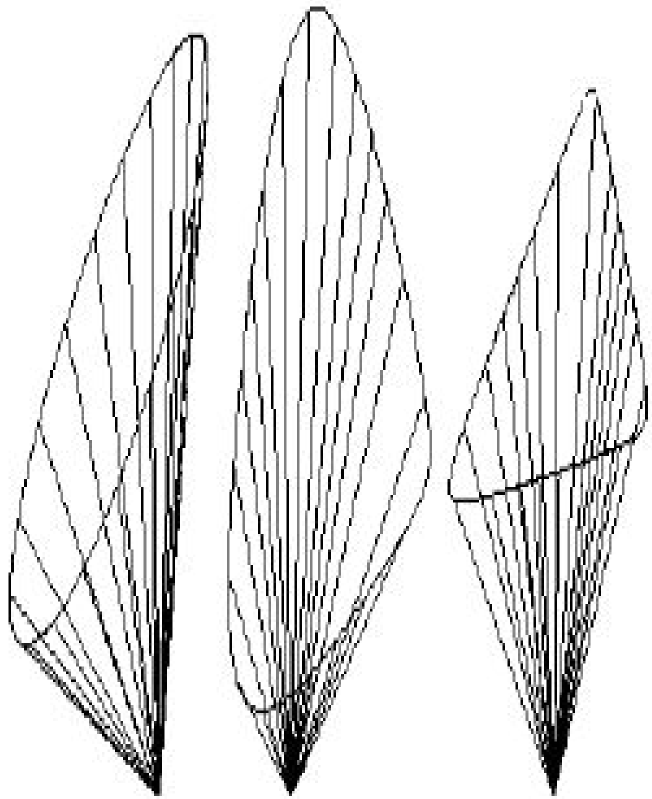
Matrix variables! -- What is the inner product?

The semidefinite cone in $\mathbb{R}^{n\times n}$ is defined as

$$\mathcal{S}^n := \{X \in \mathbb{R}^{n\times n} : X = X^T, z^T X z \geq 0\, \forall z \in \mathbb{R}^n\}$$

i.e. the matrices X are symmetric and positive semidefinite, denoted as $X \succeq 0$. The semidefinite cone is self-dual: $(\mathcal{S}^n)^* = \mathcal{S}^n$.

Optimization problems where the cones $\mathcal{C}_1$ and $\mathcal{C}_2$ are either polyhedral, second order or semidefinite cones are called *semidefinite optimization (SDO) problems.*



3 random 3D cross-sections of S^5_+

SCOPI Alumni

Semidefinite Optimization - formulation

Let A_i, $i = 1, \cdots, n$ and C, X be $n \times n$ symmetric matrices, $b, y \in \mathbb{R}^m$ and let TR$(\cdot)$ denote the trace of a matrix.

The primal-dual SDO problem is defined as

$$
\begin{array}{llll}
(SP) & \min & \mathrm{Tr}(CX) & \\
 & \text{s.t.} & \mathrm{Tr}(A_i X) - b_i & \geq 0, \ \forall i \\
 & & X & \succeq 0
\end{array}
\qquad
\begin{array}{llll}
(SD) & \max & b^T y & \\
 & \text{s.t.} & C - \sum_{i=1}^{m} A_i y_i & \succeq 0 \\
 & & y & \geq 0.
\end{array}
$$

Robust optimization, trust design

Linear matrix inequalities

Convex relaxation of nonconvex/integer problems

SCOPI Alumni

III) Software for CLO problems ➡ Use IPMs!

Software tools directly usable or via modeling systems

Classic Linear Optimization

Large scale LO problems are solved efficiently.

High performance packages, like (CPLEX, GuRoBi, XPRESS-MP, MOSEK, SAS,….) offer simplex and IPM solvers as well. Problems solved with 10^8 variables.

SOCO and SDO

Polynomial solvability established.

Traditional software is unable to handle conic constraints.

High performance packages, like (CPLEX, GuRoBi, XPRESS-MP, MOSEK)

Open Source Software: SeDuMi, SDPpack, SDPA, SDPT3, CSDP, SDPHA, etc

SOCO: Problems solved with 10^6 variables.

SDO: solved with 10^4 dimensional matrices.

IPMs for General Nonlinear Problems

Polynomial solvability established for convex problems.

Implementations for non-convex problems as well.

Specialized software is developed. (MOSEK, LOQO, IPOPT, KNITRO, etc.)

Problems solved with 10^4 dimensional matrices.

SCOPI Alumni

Mixed Integer Second Order Cone Optimization

Solve relaxation and derive Disjunctive Conic Cuts

MISOCO

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathbb{K} \\ & x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{aligned}$$

where,

- $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$
- $\mathbb{K} = \mathbb{L}^{n_1} \oplus \ldots \oplus \mathbb{L}^{n_k}$
- $\mathbb{L}^{n_i} = \{x \mid x_1 \geq \|x_{2:n_i}\|\}$
- Rows of A are linearly independent

Sample MISOCO

$$\begin{array}{llllll} \text{min:} & 3x_1 & +2x_2 & +2x_3 & +x_4 & \\ \text{s.t.:} & 9x_1 & +x_2 & +x_3 & +x_4 & = 10 \\ & & (x_1, x_2, x_3, x_4) & & \in \mathbb{L}^4 & \\ & & & x_4 & \in \mathbb{Z}. & \end{array}$$

Solve continuous relaxation.
The optimal solution is

$$x^*_{\text{soco}} = (1.36, -0.91, -0.91, -0.45),$$

and the optimal value is zero.

SCOPI Alumni

The feasible set of the sample problem

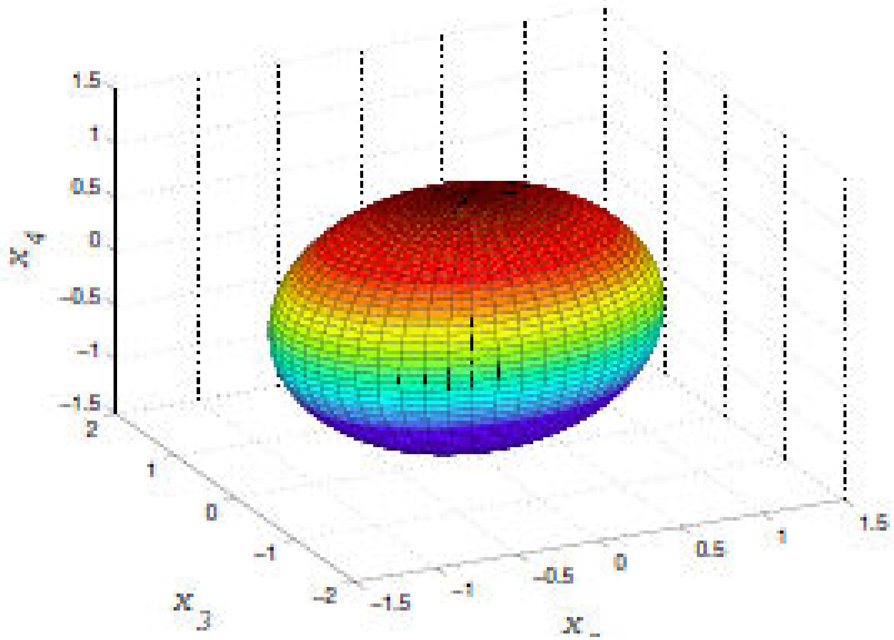
How to cut?

Reformulation of the relaxed problem

$$\text{min:} \quad \tfrac{1}{3}(10 + 5x_2 + 5x_3 + 2x_4)$$

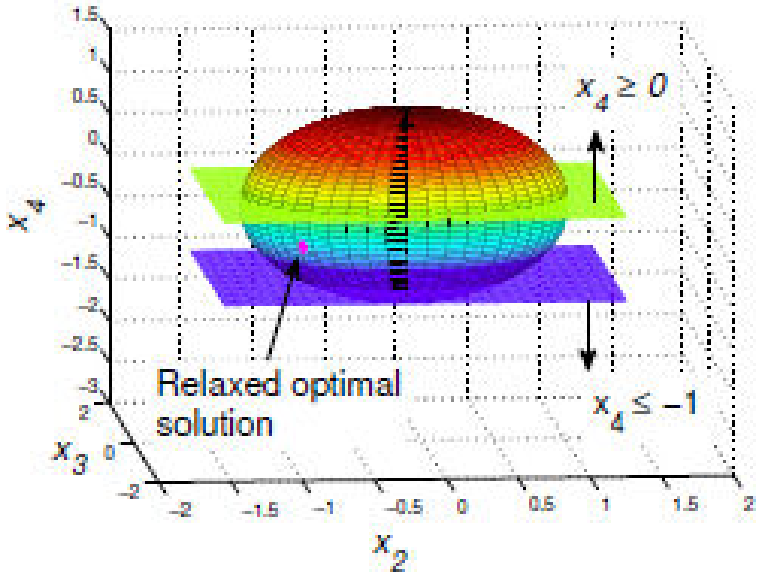
$$\text{s.t.:} \quad \begin{bmatrix} x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 8 & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & 8 & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & 8 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} - 10 \leq 0$$

$$x_4 \in \mathbb{Z}$$

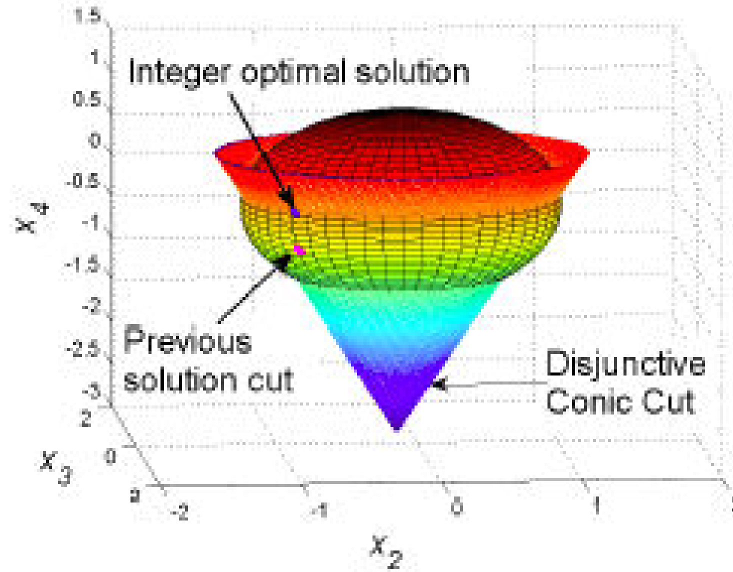


Disjunctive Conic Cut for SOCO exist & computable

The disjunction $x_4 \leq -1 \bigvee x_4 \geq 0$ is violated by x^*_{soco}



(A) Disjunction



(B) Disjunctive conic cut

An integer optimal solution is obtained after adding one cut:

$$x^*_{\text{misoco}} = x^*_{\text{soco}} = (1.32,\ -0.93,\ -0.93,\ 0.00,\ 10.06,\ -10.06,\ 0.00),$$

with an optimal objective value: $z^*_{\text{misoco}} = x^*_{\text{soco}} = 0.24$

SCOPI Alumni

Trends and advances in optimization:

Theory ⟷ Computational methods ⟷ Computers

I) Solve larger problems faster -- New algorithm paradigms
II) New model classes
III) Software for Conic Linear Optimization Problems

Industry applications

I) General and sector specific modeling/optimization tools
II) Optimization is Ubiquitous in Industry
III) Not only industry, everywhere in life

Conclusions

SCOPI Alumni

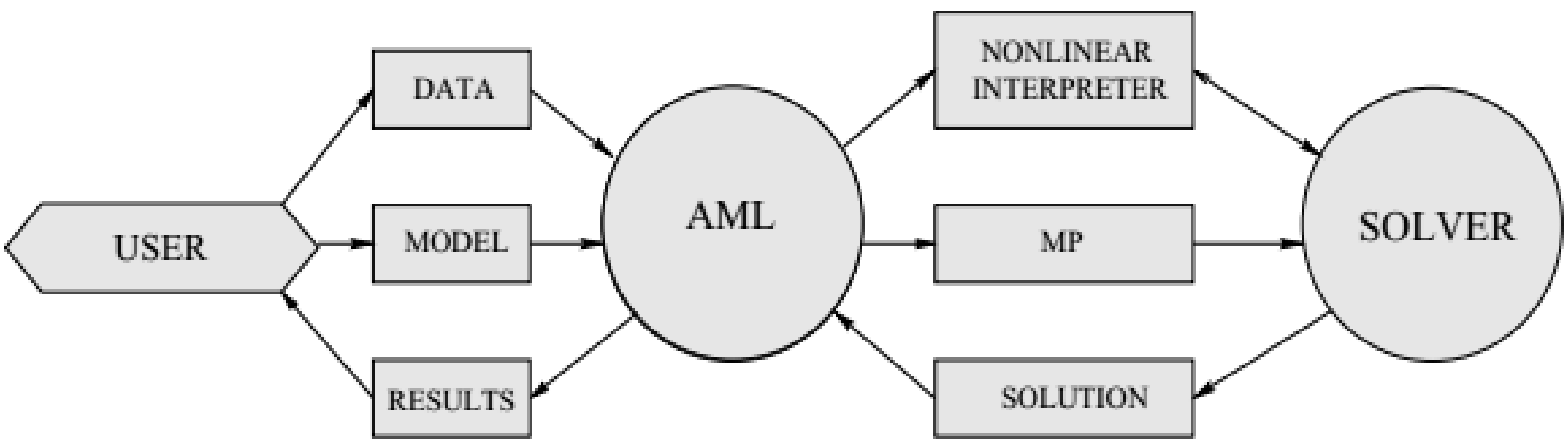
Modeling systems structure

User does not have to work directly with solver

Single model → **Access to multiple solver engines**

Nonlinear models → **Automatic/Algorithmic differentiation**
first and second order derivatives

Representation of Conic constraints



Fragniere, Gondzio (1998)

General and sector specific modeling/optimization tools

Modeling systems minimize the burden of forming and maintaining models

Note: There were no such tools in the 1960's and 70's

General purpose modeling systems

- GAMS
- AMPL
- AIMMS
- MPL, OMP
- AML, AMPL
- NEOS-Kestrel+AMLL
- **XML, GLPK, COIN-OR
- Solver vendor systems, such as
  MOSUL, FICO, NUMERICA, LGO
- LINDO, ExCEL
- SAS
- CVX
- SP/OSL, MSLiP, DECIS
- MATLAB, OCTAVE, MAPLE, Matematica

Sector specific modeling systems

- PIMS (Chemical & process industry)
- gPROMS, ASCEND (Chemical)
- CATIA (Design optimization)
- pyACDT (Airplane design)
- Genesis (design optimization)
- YALMIP (control)
- GIS (Geographical Information System)
- OptiRisk (Finance)

Model Analysis)

- ANALYZE
- MPROBE
- Visualization and Optimization

SCOPI Alumni

II) Optimization is Ubiquitous in Industry

Optimization everywhere ….

Service industries:

- Value (Supply) chain, …
- Electricity networks and markets
- Electronic marketing: Game theoretical and equilibrium models
- Data mining – machine learning
- Transportation, routing and network design
- Financial optimization, asset management, pricing
- Revenue management
- Crew assignment
- ....etc..…etc…

Engineering systems, Engineering design:

- Control systems
- Truss topology design, bridges, airplane and wing design
- Product and parts design
- Communication systems design
- Antennae design
- Nuclear reactor reloading optimization
- Battery life optimization
- ....etc..…etc…

SCOPI Alumni

III) Not only in industry, everywhere in life

- **Healthcare**
  - Operating room scheduling
  - Nurse scheduling
  - Facility Design
  - Organ transplant assignment
- **In your devices - GPS**
  - Location
  - Routing
  - Cell phone tracking
- **Government**
  - School bus routing
  - Inmate assignment in prisons
  - Homeland security
- ....etc.….etc…

- **Sciences**
  - Applied Math.
  - Optimal Control
  - Genetics
  - Chemistry (Chrystallogy)
  - Material Science
- **Medical Sciences**
  - Artificial joints and artifacts
  - Radiation therapy treatment optimization
  - MRI imaging
- **Humanities**
  - Social networks
- ....etc.….etc…

SCOPI Alumni

Conclusions

Optimization explosively grows both inside and outside of the community

- **Optimization theory made epoch making advances since 1984**
- Computing technology/capacity has grown 10^6 fold
- **Rich collection of modeling systems facilitates the use of optimization technology**
- Novel model classes are solvable by commercial software
- **Optimization is everywhere**
- Even in the era of “Big Data”, data availability, data correctness is a challenge

---

Necessity: Due to competition, financial pressure, sustainability
Possible: Due to theoretical, algorithmic, computing advances and growing number of capable people

SCOPI Alumni

SCOPI Alumni

Questions?

THANK YOU

FOR YOUR ATTENTION

www.lehigh.ed~/~tat208
Tamás Terlaky
George N. and Soteria Kledaras '87 Endowed Chair Professor.
Chair, Dept. ISE